Lyapunov Stability Theory

Applications to Stability, Robustness and Safety of Aerial Drones

Florian Pouthier

florian.pouthier@ls2n.fr

Nantes Univ., CNRS, École Centrale Nantes, LS2N, 44300 Nantes, France

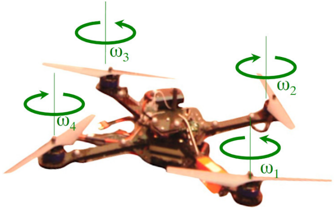
MSc Control and Robotics - Advanced Robotics (CORO IMARO) - **DRONES**

December 3, 2025 - Centrale Nantes

Challenges of Quadrotor Stability and Robustness

- Quadrotors are **nonlinear** and **underactuated** systems (6 DoF, 4 inputs)
- Recall the **quadrotor dynamics** with **nonlinearities** and **control variables**

$$\begin{cases} m\ddot{x} &= T(\sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi) \\ m\ddot{y} &= T(-\sin\phi\cos\psi + \sin\psi\sin\theta\cos\phi) \\ m\ddot{z} &= T\cos\theta\cos\phi - mg \\ I_{xx}\,\dot{p} &= qr(I_{yy} - I_{zz}) + \tau_\phi \\ I_{yy}\,\dot{q} &= pr(I_{zz} - I_{xx}) + \tau_\theta \\ I_{zz}\,\dot{r} &= pq(I_{xx} - I_{yy}) + \tau_\psi \end{cases}$$



Source: University of Pennsylvania

- **Disturbances** (wind, aerodynamic effects) and **uncertainties** also occur

- **Lecture Goals**
  - ✓ Review the **stability definitions** for (non)autonomous systems
  - ✓ Show that stability properties of linear systems no longer stand for **nonlinear systems**
  - ✓ Apply **Lyapunov's method** to analyze stability of nonlinear aerial systems
  - ✓ Get insights into the role of **input-to-state stability (ISS)** in robustness

Lecture Outline

Lecture Outline

Equilibrium Points of Autonomous Systems

- Consider the **autonomous** (no-input) time-invariant system

$$\dot{\boldsymbol{\xi}} = f(t, \boldsymbol{\xi}) \quad \text{with initial conditions } \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0$$

- $f : \mathcal{D} \to \mathbb{R}^n$ is a **locally Lipschitz map** from a domain $\mathcal{D} \subset \mathbb{R}^n$ into $\mathbb{R}^n$
- Interest: behavior of $\boldsymbol{\xi}(t)$ trajectories around equilibrium points
- **Equilibrium point** $\boldsymbol{\xi}^\star$: a trajectory that starts at $\boldsymbol{\xi}^\star$ **stays** at $\boldsymbol{\xi}^\star$ **forever** $\Rightarrow$ $f(\boldsymbol{\xi}^\star) = \boldsymbol{\xi}^\star$

Remark 1 - Non-zero equilibrium points

Without loss of generality, we consider the **equilibrium point** at the **origin**,

$$\boldsymbol{\xi}^\star = \mathbf{0} \in \mathbb{R}^n$$

Any equilibrium $\boldsymbol{\xi}^\star \neq \mathbf{0}$ can be shifted to the origin by considering the change of variables $\boldsymbol{\eta} = \boldsymbol{\xi} - \boldsymbol{\xi}^\star$.
The derivative of $\boldsymbol{\eta}$ is then given by

$$\dot{\boldsymbol{\eta}} = \dot{\boldsymbol{\xi}} = f(\boldsymbol{\eta} + \boldsymbol{\xi}^\star) \triangleq g(t, \boldsymbol{\eta}) \quad \text{where } g(\mathbf{0}) = \mathbf{0}$$

In the new variable $\boldsymbol{\eta}$, the system has equilibrium at the origin.

Stability of Equilibrium Points

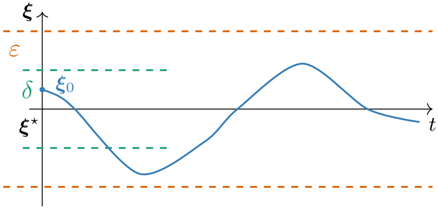
- **Stability** describes the **behavior** of $\boldsymbol{\xi}(t)$ trajectories around the equilibrium point $\boldsymbol{\xi}^\star$

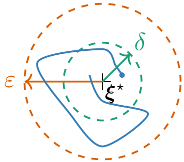
Definition 1 - Stability (in the sense of Lyapunov)

The equilibrium point $\boldsymbol{\xi}^\star = \mathbf{0}$ is stable if

$$\forall \varepsilon > 0, \quad \exists\ \delta = \delta(\varepsilon) > 0 \quad \text{such that} \quad ||\boldsymbol{\xi}_0|| < \delta \ \Rightarrow\ ||\boldsymbol{\xi}(t)|| < \varepsilon,\ \forall t \geq 0.$$

- In other words, $\boldsymbol{\xi}(t)$ trajectories **remain bounded** if initial condition $\boldsymbol{\xi}_0$ is close enough to equilibrium $\boldsymbol{\xi}^\star$





Definition 2 - Unstability

The equilibrium point $\boldsymbol{\xi}^\star = \mathbf{0}$ is unstable if it is not stable.

Attractivity of Equilibrium Points

- **Attractivity** describes the **convergence** of $\boldsymbol{\xi}(t)$ trajectories to the equilibrium point $\boldsymbol{\xi}^\star$

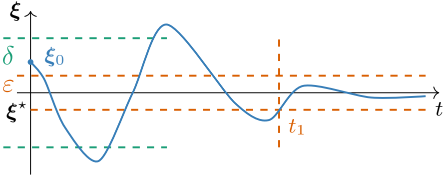
Definition 3 - Attractivity

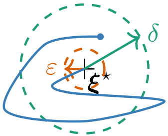
The equilibrium point $\boldsymbol{\xi}^\star = \mathbf{0}$ is **attractive** if

$$\exists \delta > 0, \quad ||\boldsymbol{\xi}_0|| < \delta \;\Rightarrow\; \lim_{t\to\infty} \boldsymbol{\xi}(t) = \mathbf{0}$$

$$\text{or} \quad \exists \delta > 0, \quad ||\boldsymbol{\xi}_0|| < \delta \;\Rightarrow\; \forall \varepsilon > 0, \quad \exists t_1 > 0 \;\text{ such that }\; \forall t > t_1, \quad ||\boldsymbol{\xi}(t)|| < \varepsilon$$

- In other words, $\boldsymbol{\xi}(t)$ trajectories **converge** to $\mathbf{0}$ for $t \to \infty$ if initial condition $\boldsymbol{\xi}_0$ is close enough to equilibrium $\boldsymbol{\xi}^\star$





Definition 4 - Asymptotic stability

The equilibrium point $\boldsymbol{\xi}^\star = \mathbf{0}$ is **asymptotically stable** if it is **stable** and **attractive**.

Stability using the Linearization Test

- Idea: Use the linear approximation of a nonlinear system at $\boldsymbol{\xi}^\star = \mathbf{0}$ to conclude on its **local stability**
- The following theorem is known as the **Lyapunov's Indirect** (first) **Method**

Theorem 1 - Local asymptotic stability - Linearization

Consider the equilibrium point $\boldsymbol{\xi}^\star = \mathbf{0}$. Calculate the Jacobian matrix of the system

$$\boldsymbol{A} = \left.\frac{\partial f(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}}\right|_{\boldsymbol{\xi}=\boldsymbol{\xi}^\star}$$

If $\text{Re}\lambda_i < 0$ for all eigenvalues of $\boldsymbol{A}$, then $\boldsymbol{\xi}^\star = \mathbf{0}$ is **asymptotically stable**.

If $\text{Re}\lambda_i > 0$ for one or more of the eigenvalues of $\boldsymbol{A}$, then $\boldsymbol{\xi}^\star = \mathbf{0}$ is **unstable**.

Remark 2 - Eigenvalues with null real parts

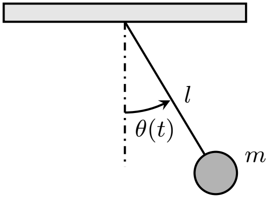
Theorem 1 does not say anything about the case when $\text{Re}\lambda_i \leq 0$ for all i, with $\text{Re}\lambda_i = 0$ for some i.

In this case, linearization fails to determine the stability of the equilibrium point.

- Drawback #1: **Theorem 1** provides only **local conclusions** on the stability!
- Drawback #2: Attractivity $\Rightarrow$ stability for linear systems, which is **not the case for nonlinear systems**!

Lecture Outline

Energy Concepts for Stability



- **Pendulum system** with state variables $x_1 = \theta$ and $x_2 = \dot{\theta}$

$$\dot{\boldsymbol{\xi}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix}$$

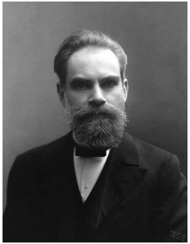
- Calculate the **pendulum energy** $E(\boldsymbol{\xi})$ = potential energy + kinetic energy.
- How $E(\boldsymbol{\xi})$ evolves in time? Draw **stability conclusions** for $k = 0$ and $k > 0$.

▶ Concept: Stability of $\boldsymbol{\xi}^\star$ can be determined by examining the derivative of E along $\boldsymbol{\xi}$ trajectories of the system

- In 1892, **A. M. Lyapunov** generalized the concept to more general functions (than energy)
- Let $V : \mathcal{D} \to \mathbb{R}$ be a $\mathcal{C}^1$ function defined in a domain $\mathcal{D} \subset \mathbb{R}^n$ that contains the origin
- The derivative of V along the system trajectories, denoted $\dot{V}(\boldsymbol{\xi})$, is given by

$$\dot{V}(\boldsymbol{\xi}) = \sum_{i=1}^{n} \frac{\partial V}{\partial \xi_i}\dot{\xi}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial \xi_i} f_i(\xi) = \frac{\partial V}{\partial \boldsymbol{\xi}} f(\boldsymbol{\xi})$$

- If $\dot{V}(\boldsymbol{\xi})$ is negative, V will decrease along the trajectory solutions of the system



Aleksandr Lyapunov in 1908

Lyapunov's Theorem of Stability

- This fundamental theorem of stability is also known as the **Lyapunov's Direct** (second) **Method** (1892)

Theorem 2 - Local (asymptotic) stability - Lyapunov's Theorem

Consider the equilibrium point $\boldsymbol{\xi}^\star = \mathbf{0}$ and a domain $\mathcal{D} \subset \mathbb{R}^n$ including $\mathbf{0}$. Let $V : \mathcal{D} \to \mathbb{R}$ be a $\mathcal{C}^1$ function such that

$$V(\mathbf{0}) = 0 \quad \text{and} \quad V(\boldsymbol{\xi}) > 0 \quad \forall \boldsymbol{\xi} \in \mathcal{D} \setminus \{0\} \quad (V \text{ is positive definite})$$

$$\dot{V}(\boldsymbol{\xi}) \leq 0 \quad \forall \boldsymbol{\xi} \in \mathcal{D} \quad (\dot{V} \text{ is negative semi-definite})$$

then $\boldsymbol{\xi}^\star = \mathbf{0}$ is **stable**. Moreover, if

$$\dot{V}(\boldsymbol{\xi}) < 0 \quad \forall \boldsymbol{\xi} \in \mathcal{D} \setminus \{0\} \quad (\dot{V} \text{ is negative definite})$$

then $\boldsymbol{\xi}^\star = \mathbf{0}$ is **asymptotically stable**.

- This theorem states the local stability on a domain $\mathcal{D}$
- Any function V satisfying the above conditions is called a **Lyapunov function**

Remark 3 - Conditions sufficiency

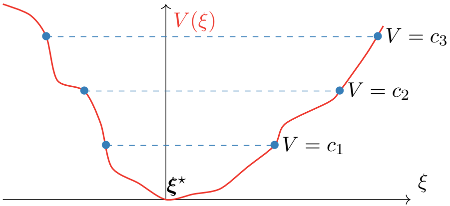
Theorem 2 only provides **sufficient** conditions for stability, not **necessary** conditions!

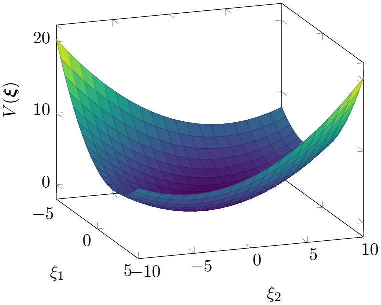
Remark 4 - Negative definiteness

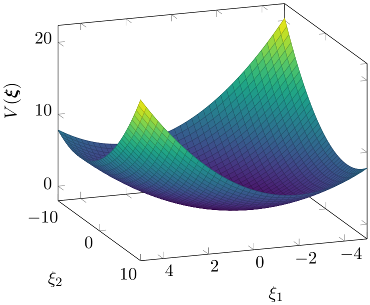
$\dot{V}$ being **negative definite** is the most difficult condition to satisfy. In many cases, $\dot{V}$ is only negative semi-definite.

Lyapunov Functions

- State domain definition: $\boldsymbol{\xi} \in \mathcal{D} \subset \mathbb{R}^n$
- $V(\mathbf{0}) = 0$, $V(\boldsymbol{\xi}) > 0 \ \forall \boldsymbol{\xi} \in \mathcal{D} \setminus \{0\}$, and $\dot{V}(\boldsymbol{\xi}) \leq 0$ (or < 0)
- **Lyapunov function** are extended to **non energetic** functions
- $V(\boldsymbol{\xi}) = c_i$ are called **level surfaces** (or Lyapunov surfaces)







Sketch of the Proof of Lyapunov's Theorem

- **Stability** is proven by definition considering the properties of V
  - Given $\epsilon > 0$, choose $r \in (0, \epsilon]$ s.t. $\mathcal{B}_r \triangleq \{\boldsymbol{\xi} \in \mathbb{R}^n \mid ||\boldsymbol{\xi}|| \leq r\} \subset \mathcal{D}$
  - Let $\alpha = \min_{||\boldsymbol{\xi}||=r} V(\boldsymbol{\xi})$. Then $\alpha > 0$ since V is pos. def.
  - Take $\beta \in (0, \alpha)$ and let $\Omega_\beta \triangleq \{\boldsymbol{\xi} \in \mathcal{B}_r \mid V(\boldsymbol{\xi}) \leq \beta\}$
  - Then $\Omega_\beta \subset \mathcal{B}_r$, and any trajectory starting in Ω_β stays in Ω_β since

$$\dot{V}(\boldsymbol{\xi}(t)) \leq \mathbf{0} \;\Rightarrow V(\boldsymbol{\xi}(t)) \leq V(\boldsymbol{\xi}(\mathbf{0})) \leq \beta,\; \forall t \geq 0$$

  - There is $\mathcal{B}_\delta$ with $\delta > 0$ s.t. $||\boldsymbol{\xi}|| \leq \delta \;\Rightarrow\; V(\boldsymbol{\xi}) < \beta$ as V cont.
  - Then $\mathcal{B}_\delta \subset \Omega_\beta \subset \mathcal{B}_r$, and

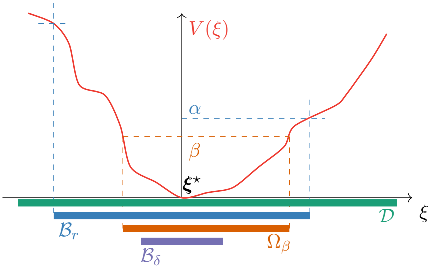
$$\boldsymbol{\xi}_0 \in \mathcal{B}_\delta \;\Rightarrow\; \boldsymbol{\xi}_0 \in \Omega_\beta \;\Rightarrow\; \boldsymbol{\xi}(t) \in \Omega_\beta \;\Rightarrow\; \boldsymbol{\xi}(t) \in \mathcal{B}_r$$

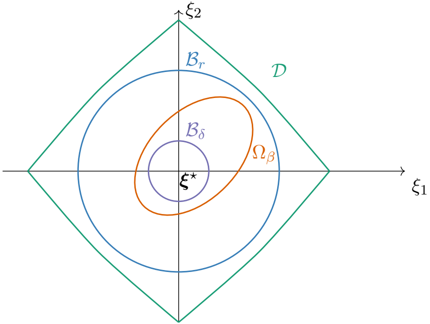
- Therefore, we have shown that $\boldsymbol{\xi}^\star = \mathbf{0}$ is stable since

$$||\boldsymbol{\xi}_0|| < \delta \;\Rightarrow\; ||\boldsymbol{\xi}(t)|| < r \leq \epsilon,\; \forall t \geq 0$$

- Prove **asymptotic stability** by using a contradiction argument

  Hint: Considering Ω_β reduces to show that $V(\boldsymbol{\xi}(t)) \to \mathbf{0}$ as $t \to \infty$





Global Asymptotic Stability

- Previous theorems considered **local** stability for a region $\mathcal{D}$
- What are the conditions to have a global property, that is for $\mathcal{D} = \mathbb{R}^n$?

Theorem 3 - Global asymptotic stability

Consider the equilibrium point $\boldsymbol{\xi}^\star = \mathbf{0}$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^1$ function such that

$$V(\mathbf{0}) = 0 \quad \text{and} \quad V(\boldsymbol{\xi}) > 0 \quad \forall \boldsymbol{\xi} \neq \mathbf{0} \qquad (V \text{ is positive definite})$$

$$||\boldsymbol{\xi}(t)|| \to +\infty \quad \Rightarrow \quad V(\boldsymbol{\xi}) \to +\infty \qquad (V \text{ is radially unbounded})$$

$$\dot{V}(\boldsymbol{\xi}) < 0 \quad \forall \boldsymbol{\xi} \neq \mathbf{0} \qquad (\dot{V} \text{ is negative definite})$$

then $\boldsymbol{\xi}^\star = \mathbf{0}$ is **globally asymptotically stable (GAS)**.

Stability of Quadrotor Rotational Dynamics

- Take the **quadrotor rotational dynamics** modeled in the previous lessons

$$\begin{cases} I_{xx}\ \dot{p} &= qr(I_{yy}-I_{zz})+\tau_\phi \\ I_{yy}\ \dot{q} &= pr(I_{zz}-I_{xx})+\tau_\theta \\ I_{zz}\ \dot{r} &= pq(I_{xx}-I_{yy})+\tau_\psi \end{cases}$$

- $\boldsymbol{\omega} \triangleq \begin{bmatrix} p & q & r \end{bmatrix}^T$ is the quadrotor **angular velocity** vector
- $\boldsymbol{\tau_B} \triangleq \begin{bmatrix} \tau_\phi & \tau_\theta & \tau_\psi \end{bmatrix}^T$ is the rotation **torque input** vector
- I_{xx}, I_{yy}, I_{zz} are the **moments of inertia** on each main axis
- Torques $\boldsymbol{\tau_G}$ due to the gyroscopic effects of the rotors are neglected

♦ Show that with $\tau_\phi = \tau_\theta = \tau_\psi = 0$, the origin $\boldsymbol{\omega}^\star = \mathbf{0}$ is **stable**.

♦ Is the origin $\boldsymbol{\omega}^\star = \mathbf{0}$ **asymptotically stable** ?

♦ Suppose the torque inputs apply the feedback control $\boldsymbol{\tau_B} = -\boldsymbol{K} \cdot \boldsymbol{\omega}$, with $\boldsymbol{K} \triangleq \begin{bmatrix} k_p & k_q & k_r \end{bmatrix}^T$ of positive coeffs.
Show that the origin of the closed-loop system is **GAS**.

Lecture Outline

LaSalle's Invariance Principle

- Is it still possible to show asymptotic stability in the case where $\dot{V}$ is only semi-negative definite?
- Idea: If no trajectory stay forever at points where $\dot{V}(\boldsymbol{\xi}) = 0$ except at $\boldsymbol{\xi}^\star = \mathbf{0}$, then $\boldsymbol{\xi}^\star$ is **asymptotically stable**.

Definition 5 - Invariant set

A set $\mathcal{M} \subset \mathbb{R}^n$ is an **invariant set** with respect to $\dot{\boldsymbol{\xi}} = f(t, \boldsymbol{\xi})$ if

$$\boldsymbol{\xi}_0 \in \mathcal{M} \;\Rightarrow\; \boldsymbol{\xi}(t) \in \mathcal{M}, \; \forall t \in \mathbb{R}$$

A set $\mathcal{M} \subset \mathbb{R}^n$ is a **positively invariant set** if

$$\boldsymbol{\xi}_0 \in \mathcal{M} \;\Rightarrow\; \boldsymbol{\xi}(t) \in \mathcal{M}, \; \forall t \geq 0$$

- If a trajectory $\boldsymbol{\xi}$ belongs to $\mathcal{M}$ at some time instant, then it belongs to $\mathcal{M}$ **for all (positive) time instants**.

Theorem 4 - LaSalle's theorem

Let $\Omega \subset \mathcal{D}$ be a compact set that is positively invariant for the autonomous system.
Let $V : \mathcal{D} \to \mathbb{R}$ be a $\mathcal{C}^1$ function such that $\dot{V}(\boldsymbol{\xi}) \leq 0$ in Ω.
Let $\mathcal{E}$ be the set of all points in Ω where $\dot{V}(\boldsymbol{\xi}) = 0$.
Let $\mathcal{M}$ be the largest invariant set in $\mathcal{E}$.
Then every trajectory $\boldsymbol{\xi}(t)$ starting in Ω approaches $\mathcal{M}$ as $t \to \infty$.

Barbashin and Krasovskii Theorems

- Showing that $\boldsymbol{\xi}(t) \to \mathbf{0}$ as $t \to \infty$ require to establish that the largest invariant set in $\mathcal{E}$ is the origin
- This is achieved by showing that no solution can stay identically in $\mathcal{E}$, except for the trivial trajectory solution $\boldsymbol{\xi}(t) = \mathbf{0}$
- These results are obtained by extending **Theorem 4** to such a case and taking $V(\boldsymbol{\xi})$ to be **positive definite**

Corollary 1 - Asymptotic stability (by invariance principle)

Consider the equilibrium point $\boldsymbol{\xi}^\star = \mathbf{0}$ and a domain $\mathcal{D} \subset \mathbb{R}^n$ including $\mathbf{0}$. Let $V : \mathcal{D} \to \mathbb{R}$ be a $\mathcal{C}^1$, positive definite function such that $\dot{V}$ is negative semi-definite in $\mathcal{D}$ ($\forall \boldsymbol{\xi} \in \mathcal{D}$, $\dot{V}(\boldsymbol{\xi}) \leq 0$).

Let $\mathcal{S} \triangleq \{\boldsymbol{\xi} \in \mathcal{D} \mid \dot{V}(\boldsymbol{\xi}) = \mathbf{0}\}$ and suppose that no solution can stay identically in $\mathcal{S}$ other than the solution $\boldsymbol{\xi}(t) = \mathbf{0}$.

Then $\boldsymbol{\xi}^\star = \mathbf{0}$ is **asymptotically stable**.

Corollary 2 - Global asymptotic stability (by invariance principle)

Consider the equilibrium point $\boldsymbol{\xi}^\star = \mathbf{0}$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^1$, radially unbounded, positive definite function such that $\dot{V}$ is negative semi-definite in $\mathbb{R}^n$ ($\forall \boldsymbol{\xi} \in \mathbb{R}^n$, $\dot{V}(\boldsymbol{\xi}) \leq 0$).

Let $\mathcal{S} \triangleq \{\boldsymbol{\xi} \in \mathbb{R}^n \mid \dot{V}(\boldsymbol{\xi}) = \mathbf{0}\}$ and suppose that no solution can stay identically in $\mathcal{S}$ other than the solution $\boldsymbol{\xi}(t) = \mathbf{0}$.

Then $\boldsymbol{\xi}^\star = \mathbf{0}$ is **GAS**.

- When $\dot{V}(\boldsymbol{\xi})$ is negative definite, $\mathcal{S} = \{\mathbf{0}\}$. Then, **Corollaries 1** and **2** coincide with **Theorems 2** and **3**, respectively.

Stability of Quadrotor Translational Dynamics

- Take the **quadrotor translational dynamics** modeled in the previous lessons

$$\begin{cases} m\ddot{x} &= T(\sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi) \\ m\ddot{y} &= T(-\sin\phi\cos\psi + \sin\psi\sin\theta\cos\phi) \\ m\ddot{z} &= T\cos\theta\cos\phi - mg \end{cases}$$

- $\boldsymbol{p} \triangleq [x \quad y \quad z]^T$ is the **CoM quadrotor position** vector
- $\boldsymbol{\Gamma} \triangleq [\phi \quad \theta \quad \psi]^T$ is the **Euler angles** vector in fixed frame
- m is the **quadrotor mass**
- g is the **gravitational acceleration** $\left(g \approx 9.81\text{m/s}^2\right)$
- T is the **total thrust** generated by the rotors

♦ Show that with $\nu_\phi = \nu_\theta = \nu_z = 0$, the hovering point $(\boldsymbol{p}, \boldsymbol{v}) = (\boldsymbol{0}, \boldsymbol{0})$ is an equilibrium.
Is this equilibrium **(asymptotically) stable** ?

♦ Take a PD-like feedback control that stabilizes the translational axes.
Show that the origin of the closed-loop system is **GAS**.

♦ Is the origin still **GAS** when taking only a proportional controller ?

Lecture Outline

Stability of Nonautonomous Systems

- Consider the **nonautonomous** (with input) time-invariant system

$$\dot{\boldsymbol{\xi}} = f(t, \boldsymbol{\xi}, \boldsymbol{\nu}) \quad \text{with initial conditions } \boldsymbol{\xi}(t_0) = \boldsymbol{\xi}_0$$

- $f : \mathcal{D} \to \mathbb{R}^n$ is a locally Lipschitz map from a domain $\mathcal{D} \subset \mathbb{R}^n$ into $\mathbb{R}^n$
- The **control input** $\boldsymbol{\nu} \in \mathbb{R}^m$ is a piecewise continuous function and is bounded

> **Assumption 1 - 0-GAS system**
> We assume that the **unforced system** $\dot{\boldsymbol{\xi}} = f(t, \boldsymbol{\xi}, \mathbf{0})$ has the **equilibrium point** $\boldsymbol{\xi}^\star = \mathbf{0}$ and is **GAS**

- How does the system behave when it is subject to a **bounded input** $\boldsymbol{\nu}$?

Definitions of Comparison Functions

- Nonautonomous systems require **uniformity** for (asymptotic) stability
- **Comparison functions** are needed to restate the stability theorems

Definition 6 - Class $\mathcal{K}$ function

A continuous function $\alpha : [0, a) \to [0, \infty)$ is **class $\mathcal{K}$** if

- it is **strictly increasing**
- it is such that $\alpha(0) = 0$

Definition 7 - Class $\mathcal{K}_\infty$ function

A continuous function $\alpha : [0, a) \to [0, \infty)$ is **class $\mathcal{K}_\infty$** if

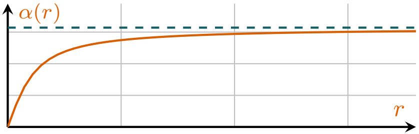
- it is **class $\mathcal{K}$**
- it is such that $a = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$

Definition 8 - Class $\mathcal{KL}$ function

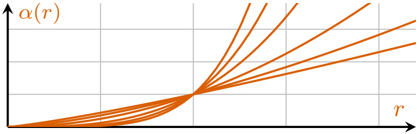
A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is **class $\mathcal{KL}$** if

- $\beta(\cdot, s)$ is **class $\mathcal{K}$**
- $\beta(r, \cdot)$ is **decreasing** and $\lim_{s\to\infty} \beta(r, s) = 0$

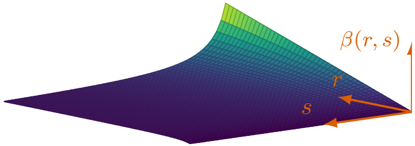
♦ What about the function $\beta(r, s) = r^k e^{-as}$, $a > 0$, $k > 1$?



$\alpha(r) = tan^{-1}(r) \to$ class $\mathcal{K}$



$\alpha(r) = r^k,\ k > 1 \to$ class $\mathcal{K}_\infty$



$\beta(r, s) = \dfrac{r}{rs + 1} \to$ class $\mathcal{KL}$

Lyapunov's Theorem with Comparison Functions

- It is possible to rewrite **Theorem 2** in terms of **comparison functions**

Theorem 5 - Local (asymptotic) stability - Comparison functions

Consider the equilibrium point $\boldsymbol{\xi}^\star = \mathbf{0}$ and a domain $\mathcal{D} \subset \mathbb{R}^n$ including $\mathbf{0}$. Let $V : \mathcal{D} \to \mathbb{R}$ be a $\mathcal{C}^1$ function such that

$$\alpha_1(||\boldsymbol{\xi}||) \leq V(\boldsymbol{\xi}) \leq \alpha_2(||\boldsymbol{\xi}||)$$

$$\dot{V}(\boldsymbol{\xi}) \leq -\alpha_3(||\boldsymbol{\xi}||)$$

Then the equilibrium point $\boldsymbol{\xi}^\star = \mathbf{0}$ is

- **stable** if α_1, α_2 are class $\mathcal{K}$ functions and $\alpha_3 \geq 0$ on $\mathcal{D}$
- **asymptotically stable** if α_1, α_2 and α_3 are class $\mathcal{K}_\infty$ functions

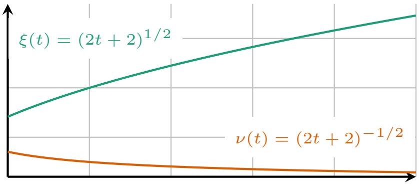
♦ Show that **Theorem 2** and **Theorem 5** are **equivalent**.

Nonautonomous Linear vs Nonlinear Systems

- Consider the **linear time-invariant system** $\dot{\boldsymbol{\xi}} = \boldsymbol{A}\boldsymbol{\xi} + \boldsymbol{B}\boldsymbol{\nu}$ where $\boldsymbol{A}$ is assumed to be **Hurwitz**
- We can write the solution trajectory $\boldsymbol{\xi}(t)$ explicitly as $\boldsymbol{\xi}(t) = e^{\boldsymbol{A}t}\boldsymbol{\xi}_0 + \displaystyle\int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{\nu}(\tau)\, d\tau$
- Since $\boldsymbol{A}$ is Hurwitz, $\exists(\lambda, \mu)$ such that $||e^{\boldsymbol{A}t}|| \leq \lambda e^{-\mu t}$, so we have

$$||\boldsymbol{\xi}(t)|| \quad \leq \quad \lambda e^{-\mu t}||\boldsymbol{\xi}_0|| + \int_0^t \lambda e^{-\mu(t-\tau)}||\boldsymbol{B}||\; ||\boldsymbol{\nu}(\tau)||\, d\tau \quad \leq \quad \lambda e^{-\mu t}||\boldsymbol{\xi}_0|| + \frac{\lambda||\boldsymbol{B}||}{\mu} \sup_{0\leq\tau\leq t} ||\boldsymbol{\nu}(\tau)||$$

- For **linear systems**, the **0-GAS property** implies that bounded inputs results in bounded state trajectories
- These properties generally do not hold for **nonlinear systems**, even under **0-GAS assumption**!
- Consider the nonlinear system $\dot{\xi} = -\xi + (\xi^2 + 1)\nu$, which is **0-GAS**
- Taking $\nu(t) = (2t+2)^{-1/2}$ and $\xi_0 = \sqrt{2}$ results in unbounded $\xi(t)$
- Even worse, the constant input $\nu = 1$ results in a finite-time explosion!



Input-to-State Stability of Nonautonomous Systems

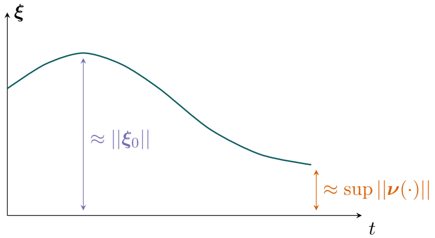
Definition 9 - Input-to-State Stability (ISS)

The system is **input-to-state stable (ISS)** if and only if it exists β of class $\mathcal{KL}$ and γ of class $\mathcal{K}$ such that

$$\forall t \geq 0, \quad ||\boldsymbol{\xi}(t)|| \leq \beta(||\boldsymbol{\xi}_0||, t) + \gamma\left(\sup_{0 \leq \tau \leq t} ||\boldsymbol{\nu}(\tau)||\right)$$

holds for all initial conditions $\boldsymbol{\xi}_0$ and all bounded inputs $\boldsymbol{\nu}(\cdot)$.

- The **ISS property** of the system means that any **bounded input** implies a **bounded state**
- The **ISS property** combines both overshoot and asymptotic behavior
- Since $\beta(||\boldsymbol{\xi}_0||, t) \to 0$ as $t \to \infty$, the state is bounded by $\sup ||\boldsymbol{\nu}(\cdot)||$
- For small t, the $\beta(||\boldsymbol{\xi}_0||, t)$ term quantifies the transient magnitude



Theorem for ISS Analysis

- The theorem for ISS property is also based on a **Lyapunov function**

Theorem 6 - Input-to-State Stability

Consider a function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ which is $\mathcal{C}^1$ and such that

$$\alpha_1(||\boldsymbol{\xi}||) \leq V(t, \boldsymbol{\xi}) \leq \alpha_2(||\boldsymbol{\xi}||)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \boldsymbol{\xi}} f(t, \boldsymbol{\xi}, \boldsymbol{\nu}) \leq -\alpha_3(||\boldsymbol{\xi}||), \quad \forall ||\boldsymbol{\xi}|| \geq \rho(||\boldsymbol{\nu}||) > 0$$

where α_1, α_2 are class $\mathcal{K}_\infty$ functions, ρ is a class $\mathcal{K}$ function and α_3 is a pos. def. function, then the system is **ISS**.

References

- **Y. Ariba**, *Analyse des Systèmes Non-Linéaires*, 4AE-SE, INSA Toulouse, 2022.
- **J. K. Hedrick and A. Girard**, *Control of Nonlinear Dynamic Systems: Theory and Applications*, 2005.
- **H. K. Khalil**, *Nonlinear Systems*, 3rd Edition, Prentice Hall, Upper Saddle River, New Jersey, 2002.
- **N. Marchand**, *Control of Nonlinear Systems*, Course of Master PSPI, Univ. Joseph Fourier, 2009-2010.
- **E. D. Sontag**, *Mathematical Control Theory*, 2nd Edition, Springer, New York, 1998.
- **E. D. Sontag**, *Input to State Stability: Basic Concepts and Results*, Lecture Notes in Mathematics, 2008.
- *Robotics: Aerial robotics course*, University of Pennsylvania. Available: `https://www.coursera.org/learn/robotics-flight`.